# Aging in ferromagnetic systems at criticality near four dimensions

Pasquale Calabrese* and Andrea Gambassi[†]

*Scuola Normale Superiore and INFN, Piazza dei Cavalieri 7, I-56126 Pisa, Italy*
(Received 19 March 2002; published 25 June 2002)

We study the off-equilibrium response and correlation functions and the corresponding fluctuation-dissipation ratio for a purely dissipative relaxation of an $O(N)$ symmetric vector model (model $A$) below its upper critical dimension. The scaling behavior of these quantities is analyzed and the associated universal functions are determined at first order in $\epsilon = 4 - d$ in the high-temperature phase and at criticality. A nontrivial limit of the fluctuation-dissipation ratio is found in the aging regime $X^{\infty} = 1/2(1 - (\epsilon/4)(N+2)/(N+8))$ $+ O(\epsilon^2)$.

## I. INTRODUCTION

The time evolution of a system relaxing towards equilibrium is characterized by two different regimes: a transient behavior with off-equilibrium evolution, for $t < t_{eq}$, and a stationary equilibrium evolution for $t > t_{eq}$. In the former a dependence of the behavior of the system on initial conditions is expected, while in the latter homogeneity of time and time reversal symmetry (at least in absence of external fields) are recovered; dynamics of fluctuations are thus described in terms of "equilibrium" dynamics, with a characteristic time scale diverging at the critical point (critical slowing down) [1].

Consider a ferromagnetic model in a disordered state for the initial time $t = 0$, and quench it at its critical temperature. During the relaxation a small external field $h$ is applied at $\mathbf{x} = 0$ after a waiting time $s$. At time $t$ the order parameter response to $h$ is given by the response function $R_{\mathbf{x}}(t,s) = \delta \langle \phi_{\mathbf{x}}(t) \rangle / \delta h(s)$, where $\phi$ is the order parameter and $\langle \cdot \rangle$ stands for the mean over stochastic dynamics. Since the system does not reach the equilibrium this function will depend both on $s$ (the "age" of the system) and $t$. This behavior is usually referred to as aging and was first noted in spin glass systems [2].

To characterize the distance from equilibrium of an aging system, evolving at a fixed temperature $T$, the fluctuation-dissipation ratio (FDR) is usually introduced [3,4]:

$$X_{\mathbf{x}}(t,s) = \frac{T R_{\mathbf{x}}(t,s)}{\partial_s C_{\mathbf{x}}(t,s)}, \quad (1.1)$$

where $C_{\mathbf{x}}(t,s) = \langle \phi_{\mathbf{x}}(t) \phi_{\mathbf{0}}(s) \rangle$ is the two-time correlation function. When the waiting time $s$ is greater than $t_{eq}$ the dynamics is homogeneous in time and the fluctuation-dissipation theorem leads to $X_{\mathbf{x}}(t,s) = 1$. This is no longer true in the aging regime [3].

In recent years, several works [2–13] have been devoted to the study of the FDR for systems exhibiting domain growth [14], or for aging systems such as glasses and spin glasses, showing that in the low-temperature phase $X(t,s)$

turns out to be a nontrivial function of its two arguments. In particular, analytical and numerical studies indicate that the limit

$$X_{\mathbf{x}=0}^{\infty} = \lim_{s \to \infty} \lim_{t \to \infty} X_{\mathbf{x}=0}(t,s) \quad (1.2)$$

vanishes throughout the low-temperature phase both for spin glass and simple ferromagnetic system [7–9,11,12].

Only recently [3,15–19] attention has been paid to the FDR, for nonequilibrium, nondisordered, and unfrustrated systems at criticality. It has been argued that the FDR (1.2) is a novel universal quantity of nonequilibrium critical dynamics. Correlation and response functions were exactly computed in the simple cases of a random walk, a free Gaussian field, and a two-dimensional $XY$ model at zero temperature and the value $X_{\mathbf{x}=0}^{\infty} = 1/2$ was found [3]. The same problem has been addressed for the $d$-dimensional spherical model [17], for the one-dimensional Ising-Glauber chain [15,16] and Monte Carlo simulation was done for the two-dimensional Ising model [17]. In all cases $X_{\mathbf{x}=0}^{\infty}$ has values ranging between 0 and 1/2 while for some urn models a different range has been found [20]. Also the scaling form for $R_{\mathbf{x}=0}(t,s)$ was rigorously established using conformal invariance [21].

In this work we investigate the nonequilibrium correlation and response functions and the associated FDR for the $O(N)$ ferromagnetic model with purely dissipative relaxation dynamics (Model $A$ of Ref. [1]) both at the critical point and in the high-temperature phase, using a field-theoretical approach (never applied so far—see concluding remarks of Ref. [17]), at first order in an $\epsilon$ expansion.

The paper is organized as follows. In Sec. II we briefly introduce the model, the scaling forms and the Gaussian (mean-field) result. In Sec. III we derive the $\epsilon$ expansion for the FDR for all values of $s$ and $t$. Finally, in Sec. IV we summarize our results and discuss some points needed for further investigation. In the Appendix, we report on some useful details on the one-loop calculation.

## II. THE MODEL

Let us consider the purely dissipative relaxation dynamics of a $N$-component field $\varphi(\mathbf{x},t)$ described by the stochastic

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*Email addresss: calabres@df.unipi.it

[†]Email address: andrea.gambassi@sns.it

Langevin equation (Model $A$ of Ref. [1])

$$\partial_t \varphi(\mathbf{x},t) = -\Omega \frac{\delta \mathcal{H}[\varphi]}{\delta \varphi(\mathbf{x},t)} + \xi(\mathbf{x},t), \qquad (2.1)$$

where $\mathcal{H}[\varphi]$ is the Landau-Ginzburg Hamiltonian

$$\mathcal{H}[\varphi] = \int d^d x \left[ \frac{1}{2}(\partial \varphi)^2 + \frac{1}{2} r_0 \varphi^2 + \frac{1}{4!} g_0 \varphi^4 \right], \qquad (2.2)$$

$\Omega$ the kinetic coefficient, and $\xi(\mathbf{x},t)$ a zero-mean stochastic Gaussian noise with

$$\langle \xi_i(\mathbf{x},t) \xi_j(\mathbf{x}',t') \rangle = 2\Omega \delta(\mathbf{x}-\mathbf{x}') \delta(t-t') \delta_{ij}. \qquad (2.3)$$

The equilibrium correlation functions, generated by the Langevin equation (2.1) and averaged over the noise $\xi$, can be obtained by means of the field-theoretical action [22,26]

$$S[\varphi,\widetilde{\varphi}] = \int dt \int d^d x \left[ \widetilde{\varphi} \frac{\partial \varphi}{\partial t} + \Omega \widetilde{\varphi} \frac{\delta \mathcal{H}[\varphi]}{\delta \varphi} - \widetilde{\varphi} \Omega \widetilde{\varphi} \right]. \qquad (2.4)$$

where $\widetilde{\varphi}(\mathbf{x},t)$ is the response field.

In Ref. [23] this formalism was extended in order to incorporate a macroscopic initial condition into Eq. (2.4): one has also to average over the initial configuration $\varphi_0(\mathbf{x}) = \varphi(\mathbf{x},t=0)$ with a weight $e^{-H_0[\varphi_0]}$ given by

$$H_0[\varphi_0] = \int d^d x \frac{\tau_0}{2} [\varphi_0(\mathbf{x}) - a(\mathbf{x})]^2. \qquad (2.5)$$

This specifies an initial state $a(\mathbf{x})$ with correlations proportional to $\tau_0^{-1}$. In this way all response and correlation functions may be obtained, following standard methods [22,26], by a perturbative expansion of the functional weight $e^{-(S[\varphi,\widetilde{\varphi}] + H_0[\varphi_0])}$.

The propagators (Gaussian two-point correlation and response functions) of the resulting theory are [23]

$$\langle \widetilde{\varphi}_i(\mathbf{q},s) \varphi_j(-\mathbf{q},t) \rangle_0 = \delta_{ij} R_q^0(t,s)$$

$$= \delta_{ij} \theta(t-s) G(t-s), \qquad (2.6)$$

$$\langle \varphi_i(\mathbf{q},s) \varphi_j(-\mathbf{q},t) \rangle_0 = \delta_{ij} C_q^0(t,s)$$

$$= \frac{\delta_{ij}}{q^2 + r_0} \left[ G(|t-s|) \right.$$

$$\left. + \left( \frac{r_0 + q^2}{\tau_0} - 1 \right) G(t+s) \right], \qquad (2.7)$$

where

$$G(t) = e^{-\Omega(q^2+r_0)t}. \qquad (2.8)$$

It has also been shown that $\tau_0^{-1}$ is irrelevant (in the renormalization group sense) for large times behavior [23].

## A. Scaling forms

When a ferromagnetic system is quenched from a disordered initial state to its critical point, the correlation length grows as $t^{1/z}$, where $z$ is the dynamical critical exponent [1]. So in momentum space, applying standard scaling arguments, all the universal functions depend only on the two products $q^z t$ and $q^z s$ [24].

In particular, we expect the scaling forms [23]

$$R_q(t,s) = q^{-2+\eta+z} \left( \frac{t}{s} \right)^\theta F_R[\Omega q^z(t-s), t/s], \qquad (2.9)$$

$$C_q(t,s) = q^{-2+\eta} \left( \frac{t}{s} \right)^\theta F_C[\Omega q^z(t-s), t/s], \qquad (2.10)$$

where $\theta$ is the initial-slip exponent of response function related to the initial-slip exponent of the magnetization $\theta'$ and to the autocorrelation exponent $\lambda_c$ [25] by the relation [23]

$$\theta' = \theta + z^{-1}(2-z-\eta) = z^{-1}(d-\lambda_c). \qquad (2.11)$$

The functions $F_R(y,x)$ and $F_C(y,x)$ are universal apart from the normalizations for small arguments. These functions are regular functions of both arguments, and for large $x$ they behave as

$$F_R(y,x) = F_R^\infty(y) + O(x^{-1}), \qquad (2.12)$$

$$F_C(y,x) = \frac{F_C^\infty(y)}{x} + O(x^{-2}), \qquad (2.13)$$

so that, for $s \to 0$, these scaling forms reduce to the ones of Ref. [23]. We would also mention that transforming Eqs. (2.9) and (2.10) in the real $\mathbf{x}$ space (to this end we have to assume that $F_R$ and $F_C$ are rapidly decreasing functions of $y$ for $y \to \infty$) and setting $\mathbf{x}=0$ one easily obtain the scaling forms reported in Refs. [17,21]. They also reduce to the equilibrium ones [1] when $t \sim s \gg 1$.

Let us introduce [27]

$$\mathcal{X}_\mathbf{q} = \frac{\Omega R_\mathbf{q}(t,s)}{\partial_s C_\mathbf{q}(t,s)}. \qquad (2.14)$$

then, from the scaling forms above, it is simple to show that, assuming $F_C(y,x) = O(y)$, $\forall x$, the FDR may be written as

$$\mathcal{X}_{\mathbf{q}=0}^\infty = \lim_{s \to \infty} \lim_{t \to \infty} \mathcal{X}_{\mathbf{q}=0}(t,s) = (1-\theta)^{-1} \frac{F_R^\infty(0)}{F_C'^\infty(0)}. \qquad (2.15)$$

## B. Gaussian FDR

For the Gaussian model we know exactly the response and correlations functions, so we can evaluate the FDR (in Ref. [3] the related quantity $X_\mathbf{x}$ has been considered, see Sec. IV). From Eqs. (2.6) and (2.7) and definition (2.14) we have

$$\mathcal{X}_q^0(t,s) = \left(\frac{\partial_s C_q^0}{\Omega R_q^0}\right)^{-1}$$

$$= (1 + e^{-2\Omega(q^2+r_0)s} + \Omega q^2 \tau_0^{-1} e^{-2\Omega(q^2+r_0)s})^{-1}.$$

$$(2.16)$$

If the theory is off-critical ($r_0 \neq 0$) the limit of this ratio for $s \rightarrow \infty$ is 1 for all values of $q$, in agreement with the idea that in the high-temperature phase all modes have a finite equilibration time, so that equilibrium is recovered and as a consequence the fluctuation-dissipation theorem holds. For the critical theory, i.e., $r_0 \propto T - T_c = 0$, if $q \neq 0$ the limit ratio is again equal to 1, whereas for $q = 0$ we have $\mathcal{X}_{q=0}^0(t,s) = 1/2$. This analysis clearly shows that the only mode characterized by aging, i.e., which "does not relax" to the equilibrium, is the zero mode in the critical limit.

### III. ONE-LOOP FDR

The aim of this section is the computation of the nonequilibrium response and correlation functions for the purely dissipative dynamics of the $N$-vector model at one-loop order. We use here the method of renormalized field theory in the minimal subtraction scheme. The breaking of homogeneity in time gives rise to some technical problems in the renormalization procedure in terms of one-particle irreducible correlation functions (see Ref. [23] and references therein) so our computation is done in terms of connected functions.

At one-loop order we have to evaluate, taking also into account causality [22], the three Feynman diagrams in Fig. 1, one for the response function and two for the correlation one. In terms of these diagrams, we have

$$R_q(t,s) = R_q^0(t,s) - \frac{N+2}{6}\Omega g_0(a) + O(g_0^2),$$

$$(3.1)$$

$$C_q(t,s) = C_q^0(t,s) - \frac{N+2}{6}\Omega g_0[(b) + (c)] + O(g_0^2).$$

In order to evaluate the FDR at criticality we have to set in this perturbative expansion $r_0 = 0$ (massless theory). We also set $\tau_0^{-1} = 0$, since it is an irrelevant variable [23], and $\Omega = 1$ to lighten the notations. The first step in the calculation of the diagrams is the evaluations of the critical "bubble" $B_c(t)$, i.e., their common one-particle irreducible part. We have, in generic dimension $d$,

$$B_c(t) = \int \frac{d^d q}{(2\pi)^d} C_q^0(t,t)$$

$$= -\frac{1}{d/2-1}\frac{(2t)^{1-d/2}}{(4\pi)^{d/2}}$$

$$= -N_d \frac{\Gamma(d/2-1)}{2^{d/2}}t^{1-d/2}, \qquad (3.2)$$

where $N_d = 2/[(4\pi)^{d/2}\Gamma(d/2)]$. Note that the equilibrium contribution to $B_c(t)$ is zero for $d > 2$.
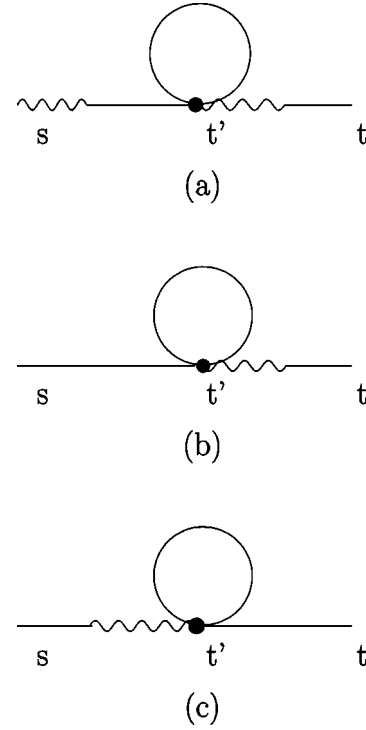


FIG. 1. Feynman diagrams contributing to the one-loop response (a) and correlation function (b) and (c). Response functions are drawn as wavy-normal lines, whereas correlators are normal lines. A wavy line is attached to the response field and a normal one to the order parameter.

Let us consider $t > s$ in the following. We may write, for generic $r_0 \geq 0$

$$(a) = \int_0^\infty dt' R_q^0(t,t')B(t')R_q^0(t',s),$$

$$(b) = \int_0^\infty dt' R_q^0(t,t')B(t')C_q^0(t',s), \qquad (3.3)$$

$$(c) = \int_0^\infty dt' R_q^0(s,t')B(t')C_q^0(t',t),$$

where we set, now, $r_0 = 0$ in $R_q^0$ and $C_q^0$, and the bubble $B(t)$ is replaced with its critical expression $B_c(t)$.

Performing the integration and expanding in powers of $\epsilon$ we find for the response function

$$R_q(t,s) = G(t-s)\left(1 + \tilde{g}_0 \frac{N+2}{24}\ln\frac{t}{s}\right) + O(\epsilon\tilde{g}_0, \tilde{g}_0^2),$$

$$(3.4)$$

and for the correlation function

$$C_q(t,s) = \frac{G(t-s) - G(t+s)}{q^2} \left( 1 + \tilde{g}_0 \frac{N+2}{24} \ln \frac{t}{s} \right)$$
$$- \tilde{g}_0 \frac{N+2}{24} \frac{G(t+s)}{q^2} f(2q^2 s) + O(\epsilon \tilde{g}_0, \tilde{g}_0^2),$$

(3.5)

where

$$f(v) = 2 \left[ \int_0^v d\xi \ln \xi e^\xi + (1 - e^v) \ln v \right],$$

(3.6)

and $\tilde{g}_0 = N_d g_0$. Note that $f(0) = 0$, $f'(0) = -2$, and $f(v)$ has the following asymptotic expansion, for $v \gg 1$:

$$f(v) = -2 \frac{e^v}{v} \left( 1 + \frac{1}{v} + \frac{2}{v^2} + \cdots + \frac{k!}{v^k} + \cdots \right).$$

(3.7)

In order to obtain the critical functions we have to set the renormalized coupling equal to its fixed point value. At first order in $\epsilon$ [26],

$$\tilde{g}_0 = \tilde{g}^* = \frac{6}{N+8} \epsilon + O(\epsilon^2).$$

(3.8)

Finally, we get $[P_N = (N+2)/(N+8)]$

$$R_q(t,s) = G(t-s) \left( 1 + \epsilon \frac{P_N}{4} \ln \frac{t}{s} \right) + O(\epsilon^2),$$

(3.9)

$$C_q(t,s) = \frac{G(t-s) - G(t+s)}{q^2} \left( 1 + \epsilon \frac{P_N}{4} \ln \frac{t}{s} \right)$$
$$- \epsilon \frac{P_N}{4} \frac{G(t+s)}{q^2} f(2q^2 s) + O(\epsilon^2). \quad (3.10)$$

These are fully compatible with the scaling form given in the preceding section, with

$$F_R(y,x) = e^{-y} + O(\epsilon^2)$$

(3.11)

and

$$F_C(y,x) = e^{-y} - \left[ 1 + \epsilon \frac{P_N}{4} f\left( \frac{2y}{x-1} \right) \right] e^{-y[(x+1)/(x-1)]}$$
$$+ O(\epsilon^2).$$

(3.12)

In particular, we recognize the exponent $\theta = P_N \epsilon/4 + O(\epsilon^2)$ in agreement with Ref. [23], $z = 2 + O(\epsilon^2)$, $\eta = O(\epsilon^2)$, as expected, and that $F_R(y,x)$ is not affected by $O(\epsilon)$ corrections. It is also easy to find that

$$F_C^\infty(y) = 2y \left( 1 + \epsilon \frac{P_N}{2} \right) e^{-y} + O(\epsilon^2).$$

(3.13)

Computing the derivative with respect to $s$ of the two-time correlation function and taking its ratio with the response function, we have

$$\mathcal{X}_q^{-1}(s) = 1 + e^{-2q^2 s} - \frac{P_N \epsilon}{4} e^{-2q^2 s} \left[ \frac{e^{2q^2 s} - 1}{q^2 s} - f(2q^2 s) \right.$$
$$\left. + 2f'(2q^2 s) \right] + O(\epsilon^2).$$

(3.14)

Note that, at least at this order, the result is independent of the observation time $t$. Using the large $v$ behavior of $f(v)$, cf. Eq. (3.7), we find that the limit of the FDR for $s \to \infty$ is equal to 1 for all $q \neq 0$. Instead, for $q = 0$ we have [using Eq. (3.6)]

$$\mathcal{X}_{q=0}^\infty = \frac{1}{2} \left( 1 - \epsilon \frac{P_N}{4} \right) + O(\epsilon^2),$$

(3.15)

in agreement with Eq. (2.15) and with the scaling forms (3.11) and (3.13).

Taking into account the effect of the mass $r_0$ (deviation from critical temperature) in the previous computations, one obtains for the noncritical bubble (contributing to the mass renormalization)

$$B(t) = N_d \left[ \frac{\pi}{2 \sin d\pi/2} - \frac{1}{2} \Gamma(d/2) \Gamma(1 - d/2, 2r_0 t) \right] r_0^{d/2 - 1},$$

(3.16)

where $\Gamma(x,y)$ is the incomplete $\Gamma$ function [28]. Using this expression it is possible to determine, as previously done, correlation and response functions. We report on the basic formulas in the Appendix. The final result is obtained computing the ratio $\mathcal{X}_q$ in terms of the renormalized parameters of the theory. It is then trivial, but algebraically cumbersome, to show that $\mathcal{X}_q^\infty$ is equal to 1 for all $q$ in the high-temperature phase.

## IV. DISCUSSION

In this work we considered the off-equilibrium properties of the purely dissipative relaxational dynamics of an $N$-vector model in the framework of field-theoretical $\epsilon$ expansion. We computed at first order in $\epsilon$ the FDR, as defined in Eq. (2.14) as a function of the waiting time $s$ and of the observation time $t$ both at criticality and in the high-temperature phase. The main result is that the ratio $\mathcal{X}_q^\infty$ is always 1 unless at criticality for $q = 0$, when it takes the value

$$\mathcal{X}_{\mathbf{q}=0}^\infty = \frac{1}{2} \left( 1 - \frac{\epsilon}{4} \frac{N+2}{N+8} \right) + O(\epsilon^2).$$

(4.1)

To compare our result with some particular limit considered in the literature [3,17], we have to relate this quantity to the analog in the real $\mathbf{x}$ space. The following heuristic argument may be useful to realize that the two ratios are exactly equal, i.e.,

$$X_{\mathbf{x}=0}^\infty = \mathcal{X}_{\mathbf{q}=0}^\infty.$$

(4.2)

We may rewrite the FDR in real $\mathbf{x}$ space as a mean value of that in momentum space with a weight given by $R_{\mathbf{q}}$:

$$X_{\mathbf{x}=0}^{-1} \equiv \frac{\displaystyle\int d^d q\, \partial_s C_{\mathbf{q}}(t,s)}{T\displaystyle\int d^d q\, R_{\mathbf{q}}(t,s)}$$

$$= \frac{\displaystyle\int d^d q\, R_{\mathbf{q}}(t,s)[\partial_s C_{\mathbf{q}}(t,s)/T R_{\mathbf{q}}(t,s)]}{\displaystyle\int d^d q\, R_{\mathbf{q}}(t,s)}$$

$$= \langle \mathcal{X}_{\mathbf{q}}^{-1} \rangle_{R_{\mathbf{q}}}. \tag{4.3}$$

Now, since we expect $R_{\mathbf{q}} \propto e^{-q^2(t-s)}$, in the limit $s,t \to \infty$ (in the right order), $X_{\mathbf{x}=0}^{-1}$ will take contributions only for the $q = 0$ mode, i.e., apart from a normalization, the weight function $R_{\mathbf{q}}$ is a $\delta(\mathbf{q})$. However, we note that at the first order in $\epsilon$ the equality (4.2) is identically satisfied by our result since $R_{\mathbf{q}} \propto e^{-q^2(t-s)}$, cf. Eq. (3.4).

In the limit $N \to \infty$, Eq. (4.1) reduces to $X^\infty = 1/2 - \epsilon/8 + O(\epsilon^2)$ that is the same as the expansion of the result for the spherical model near four dimensions [17].

A Monte Carlo simulation of the two-dimensional Ising model gave for the FDR the value $X^\infty = 0.26(1)$ [17], qualitatively in agreement with our result for $\epsilon = 2$, $X^\infty = 5/12 < 1/2$. To have a reliable quantitative prediction the knowledge of higher loop contributions is required.

Setting $\epsilon = 1$ for $N = 1$, one obtains $11/24 \sim 0.46$. This number, that is a rough theoretical estimate of the actual three-dimensional value, is indeed in good agreement with a preliminary Monte Carlo result reported in Ref. [17], $X^\infty \sim 0.40$.

This work may be easily extended to more realistic models than those previously considered in literature, contributing to the understanding of out-of-equilibrium dynamic phenomena, currently under intensive investigation, by means of the powerful tools of perturbative field theory.

## ACKNOWLEDGMENTS

## APPENDIX: DETAILS OF COMPUTATIONS

We summarize here the main analytical results useful for the computation of correlation and response functions for $r_0 \geqslant 0$ at one loop. Again one has to perform all the needed integrations over the times, as in Eq. (3.3) with the free field correlator and response function (2.6) and (2.7). At variance with the critical theory a renormalization of the parameter $r_0$ is now required to cancel dimensional poles both in $R_q$ and $C_q$.

Let us introduce the function

$$Y(t) \equiv \int_0^t d\tau\, B(\tau) \tag{A1}$$

and

$$W(t) \equiv \int_0^t d\tau\, G(-2\tau) B(\tau), \tag{A2}$$

where $G(t)$ and $B(t)$ are given in Eqs. (2.8) and (3.16), respectively. In terms of $Y$ and $W$ we obtain (for $t>s$ and $\tau_0^{-1}=0$, $\Omega=1$)

$$(a) = G(t-s)[Y(t)-Y(s)] \tag{A3}$$

and

$$(b)+(c) = \frac{1}{q^2+r_0}\{G(t-s)[Y(t)-Y(s)]$$
$$- G(t+s)[Y(t)+Y(s)]+2G(t+s)W(s)\}. \tag{A4}$$

In the following with $Y$ and $W$ we mean also their analytic continuation in $d$.

An explicit computation leads to

$$Y(t) = \frac{r_0^{d/2-2}}{2(4\pi)^{d/2}}\{(2r_0 t+d/2-1)$$
$$\times[\Gamma(1-d/2)-\Gamma(1-d/2,2r_0 t)]$$
$$+ (2r_0 t)^{1-d/2}e^{-2r_0 t}\} \tag{A5}$$

and

$$W(t) = \frac{1}{2(4\pi)^{d/2}}\frac{r_0^{d/2-1}}{q^2+r_0}$$
$$\times\{G(-2t)[\Gamma(1-d/2)-\Gamma(1-d/2,2r_0 t)]$$
$$- (q^2/r_0)^{d/2-1}\Delta(1-d/2,2q^2 t)\}, \tag{A6}$$

where we introduced

$$\Delta(v,w) \equiv \int_0^w d\tau\, \tau^{v-1}e^\tau \tag{A7}$$

(for $v \leqslant 0$ its analytic continuation has to be considered).

Expanding Eqs. (A5) and (A6) in $\epsilon = 4 - d$, we obtain

$$2(4\pi)^{d/2}Y(t) = -\frac{2}{\epsilon}(2r_0 t+1)-(2r_0 t+1)$$
$$\times[\gamma(2r_0 t)-\ln r_0]+1+\frac{e^{-2r_0 t}}{2r_0 t}+O(\epsilon) \tag{A8}$$

and

$$2(4\pi)^{d/2}(q^2+r_0)W(t)$$
$$= -\frac{2}{\epsilon}[r_0 G(-2t)+q^2]+q^2[\ln q^2 - \delta(2q^2t)]$$
$$- r_0 G(-2t)[\gamma(2r_0 t)-\ln r_0]+O(\epsilon), \quad (A9)$$

where

$$\gamma(v)\equiv 1+e^{-v}\left(\ln v+\frac{1}{v}\right)+\int_0^v d\xi \ln\xi e^{-\xi} \quad (A10)$$

and

$$\delta(v)\equiv 1+e^{v}\left(\ln v-\frac{1}{v}\right)-\int_0^v d\xi \ln\xi e^{\xi}. \quad (A11)$$

It is easy to find that $f(v)$ in Eq. (3.6) is related to $\delta(v)$ by

$$f(v)=2\left[1+\ln v-\delta(v)-\frac{e^v}{v}\right]. \quad (A12)$$

Plugging Eqs. (A8) and (A9) into Eqs. (A3) and (A4) and then into Eq. (3.1), it is easy to realize that to cancel dimensional poles both in $R_q(t,s)$ and $C_q(t,s)$ a renormalization of the bare mass $r_0$ is sufficient (at least for the case $\tau_0^{-1}=0$ we are considering);

$$r_0=Z_r r \quad \text{with} \quad Z_r=1+\frac{N+2}{3}\frac{g_0}{(4\pi)^{d/2}}\frac{1}{\epsilon}+O(g_0^2),$$
$$(A13)$$

which is in agreement with what one would expect from the corresponding static field theory (see, for instance, Ref. [26]). All the previously stated results easily follow from explicit expressions given above.

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